Stochastic simulations of high-Reynolds-number turbulence in two dimensions

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The dynamics of two-dimensional (2D) turbulence is studied by a stochastic simulation method. The latter is based on a representation of the random vorticity field and stream function by a multivariate stochastic process defined by a discrete master equation. It is demonstrated that in the continuum limit the complete hierarchy of coupled moment equations for the statistical formulation of the 2D Navier-Stokes equation is obtained. The probabilistic time evolution leads to random stresses, which can be traced to thermal fluctuations and allow one to disentangle hydrodynamic and thermodynamic degrees of freedom by some kind of renormalization procedure. The stochastic simulations at a large-scale Reynolds number of 2.5×10^5 clearly show the existence of a k^{-3} power law, where k is the wave number, in the inertial range of the energy spectrum, as is predicted by the Kraichnan-Batchelor theory.

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I. INTRODUCTION

Many theoretical and numerical approaches to the problem of turbulence are based on a statistical description of the flow field. Within the usual description of statistical fluid mechanics [1], one considers an ensemble of systems each member of which evolves in time according to the Navier-Stokes equation. This procedure yields a special type of Markov process [2] which is distinguished by the fact that the differential form of the corresponding Chapman-Kolmogorov equation is the Liouville equation describing the phase flow of the Navier-Stokes equation.

In this paper, we construct a stochastic formulation of the problem of turbulence which is based on a multivariate master equation governing the dynamics of the random vorticity field ω and the stream function ψ defined on a discrete lattice. This master equation represents a more general differential Chapman-Kolmogorov equation than the Liouville equation for the special Markov process usually considered in statistical fluid mechanics: Within our approach, in addition to random initial conditions, the time evolution of the process (ω, ψ) itself is also intrinsically probabilistic [3]. This means that single realizations of the Markov process defined by our master equation do not obey the deterministic (discretized) Navier-Stokes equation. However, as will be shown below, the moments of the stochastic process do obey the infinite hierarchy of moment equations of the turbulence.

It is important to note that the master equation contains an additional dimensionless parameter α which is not present in the Navier-Stokes equation. This parameter measures the size of the intrinsic fluctuations and will be shown to be related to the temperature of the fluid. Deriving the equation of motion for the generating functional of the stochastic process (ω, ψ), we demonstrate that performing the limit $\alpha \longrightarrow 0$ and the limit of continuous space, the Hopf functional equation [4] of the two-dimensional turbulence is recovered. Thus, in the limit of vanishing intrinsic fluctuations the master equation reduces to the Liouville equation of statistical fluid dynamics.

Furthermore, we show that for finite values of α, i.e., for finite values of the temperature, the stochastic formulation yields an additional term in the time-evolution equation for the generating functional which leads to random stresses in the equation for the two-point vorticity correlation function. These stresses can be interpreted as spontaneous vorticity fluxes which ensure equipartition of vorticity in the final stage of decay. By means of a kind of fluctuation-dissipation theorem the strength of these stresses can be related to the enstrophy temperature of a canonical ensemble based on the enstrophy as the constant of motion.

The work to be presented in this paper is based on a master equation representation of the two-dimensional Navier-Stokes turbulence which has been developed in Ref. [5]. We construct a slightly modified master equation which has the advantage of leading to stochastic simulation algorithms which are even more efficient. On the basis of this master equation, we then perform some stochastic simulations of the two-dimensional turbulence for very high Reynolds numbers. Our simulation results clearly show the existence of a k^{-3} power law in the inertial range which is precisely the prediction of the Kraichnan-Batchelor theory [6,7] of the vorticity cascade in 2D turbulence.

The paper is organized as follows. In Sec. II, we define the stochastic process representing the random vorticity field and the stream function by means of a multivariate master equation. Then we derive the equation of motion for the generating function pertaining to the master equation and show that it leads to the Hopf functional equation of the 2D turbulence in the continuum limit. In Sec. III, we perform stochastic simulations of the master equation for high Reynolds numbers and compare our results with other analytical and numerical theories of 2D-turbulence. Finally, in Sec. IV we draw our conclusions.

II. DEFINITION OF THE MASTER EQUATION

We consider the Navier-Stokes equation on the two-dimensional plane with coordinates $\vec{x} \equiv (x, y)$ for an incompressible fluid with kinematic viscosity ν. Employing the stream function formulation [8], we represent the velocity field $\vec{v} \equiv (u, v)$ by the scalar stream function $\psi(\vec{x}, t)$ as $u = \partial_y \psi$, $v = -\partial_x \psi$. The vorticity field $\omega(\vec{x}, t)$ is defined by $\omega = \partial_x v - \partial_y u$, whereas the connection between the stream function and the vorticity appears as the constraint $\Delta\psi + \omega = 0$, where Δ denotes the Laplace operator in two dimensions. The Navier-Stokes equation for an incompressible fluid can then be written in terms of the vorticity ω as

$$\frac{\partial \omega}{\partial t} = \nu \Delta \omega + \frac{\partial}{\partial x}\left(\psi \frac{\partial \omega}{\partial y}\right) - \frac{\partial}{\partial y}\left(\psi \frac{\partial \omega}{\partial x}\right) \quad . \tag{1}$$

To be specific we assume periodic boundary conditions on a square Q with side length L.

In order to construct the multivariate master equation which governs the probabilistic dynamics of our formulation of statistical fluid mechanics, we first specify the discrete phase space, that is the set of states of the fluid. To this end, we partition the position space, i.e., the square Q, into small square cells (of area δl^2) labeled by two integers (λ, μ). Thus, we write $\vec{x}_{\lambda\mu} \equiv (x_{\lambda\mu}, y_{\lambda\mu}) = (\lambda\delta l, \mu\delta l)$ for the discrete position vector, where λ and μ denote integers which run from 0 to n, and $\delta l = L/(n+1)$. Furthermore, for the sake of a compact notation, we introduce the discrete operators d_1, d_2, and D which replace the partial differential operators $\partial/\partial x$, $\partial/\partial y$, and the Laplacian Δ and which are defined by

$$\begin{aligned} d_1 f_{\lambda\mu} &:= \frac{f_{\lambda+1,\mu} - f_{\lambda-1,\mu}}{2\delta l} \quad , \\ d_2 f_{\lambda\mu} &:= \frac{f_{\lambda,\mu+1} - f_{\lambda,\mu-1}}{2\delta l} \quad , \\ D f_{\lambda\mu} &:= \frac{f_{\lambda+1,\mu} + f_{\lambda-1,\mu} + f_{\lambda,\mu+1} + f_{\lambda,\mu-1} - 4 f_{\lambda\mu}}{\delta l^2} \quad , \end{aligned} \tag{2}$$

where $f_{\lambda\mu}$ denotes an arbitrary function on the discrete grid introduced above.

The phase space Γ underlying our stochastic formulation is given by the space of all $2(n+1)^2$-dimensional arrays

$$(\omega, \psi) \equiv (\{\omega_{\lambda\mu}\}, \{\psi_{\lambda\mu}\}) \quad , \tag{3}$$

and can thus formally be written as

$$\Gamma = \{(\omega, \psi) \mid \omega_{\lambda\mu}, \psi_{\lambda\mu} \in \mathbb{R}\} \quad . \tag{4}$$

Introducing the normalized joint probability distribution $P = P(\omega, \psi, t)$ on the phase space Γ, the array (ω, ψ) becomes a multivariate stochastic process. With the help of P expectation values of arbitrary functions, $\mathcal{F} = \mathcal{F}(\omega, \psi)$ of the stochastic variables are determined by

$$\langle \mathcal{F} \rangle = \int \mathcal{D}\omega\, \mathcal{D}\psi\; \mathcal{F}(\omega, \psi)\; P(\omega, \psi, t) \quad , \tag{5}$$

where the integral denotes a $2(n+1)^2$ fold integral over the stochastic variables:

$$\int \mathcal{D}\omega\, \mathcal{D}\psi \equiv \int \prod_{\lambda\mu} d\omega_{\lambda\mu}\, d\psi_{\lambda\mu} \quad . \tag{6}$$

Invoking now the Markov property [2], the stochastic process (ω, ψ) is completely defined by a master equation for the joint probability distribution P which can be written in the compact form

$$\frac{\partial}{\partial t} P(\omega, \psi, t) = \mathcal{A} P(\omega, \psi, t) \quad . \tag{7}$$

Here, the time evolution operator $\mathcal{A}$ represents a linear operator which acts upon functions of the stochastic variables,

$$F(\omega, \psi) \longmapsto (\mathcal{A}F)(\omega, \psi) \quad . \tag{8}$$

In order to define this operator, we introduce a special class of linear operators [9]. Consider a map

$$b : \Gamma \longrightarrow \Gamma \;, \quad (\omega, \psi) \longmapsto b(\omega, \psi) \quad , \tag{9}$$

which is assumed to be close to the identity. To any such map there corresponds an operator $\mathfrak{b}$ defined by

$$\mathfrak{b} F(\omega, \psi) \equiv F(b^{-1}(\omega, \psi)) \quad . \tag{10}$$

In the following, we use the convention to denote maps of the type in Eq. (9) by small letters whereas the corresponding linear operator in the space of functions of the stochastic variables will be denoted by the same letter in german letters.

The time evolution operator $\mathcal{A}$ is decomposed as

$$\mathcal{A} = \mathcal{A}_d + \mathcal{A}_c + \mathcal{A}_p \quad . \tag{11}$$

The operator $\mathcal{A}_d$ which models the viscosity term in the Navier-Stokes equation is defined by

$$\mathcal{A}_d = \frac{\nu}{\alpha \delta l^2} \sum_{\lambda\mu} \sum_{i=1}^{4} \left(\det[(b^i_{\lambda\mu})^{-1}] \mathfrak{b}^i_{\lambda\mu} - I \right) \quad . \tag{12}$$

Here I denotes the identity operator and $\mathfrak{b}^i_{\lambda\mu}$ is the operator corresponding to the linear map defined by

$$b^i_{\lambda\mu} : \begin{cases} \omega_{\lambda\mu} & \longmapsto \omega_{\lambda\mu} - \alpha\omega_{\lambda\mu}, \\ \omega_{\lambda_i,\mu_i} & \longmapsto \omega_{\lambda_i,\mu_i} + \alpha\omega_{\lambda\mu}. \end{cases} \tag{13}$$

In this equation, (λ_i, μ_i) is one of the four nearest-neighbor cells of (λ, μ) and it is understood in Eq. (13) that all other variables that are not specified are left unchanged under the action of $b^i_{\lambda\mu}$. Furthermore, α is a small number (dimensionless and positive) the physical meaning of which is explained below. The nonlinear convection term of the vorticity equation is represented by the operator

$$\mathcal{A}_c = \frac{1}{2\gamma\delta l} \sum_{\lambda\mu} \sum_{i=1}^{2} \left(\det[(c^i_{\lambda\mu})^{-1}] \mathfrak{c}^i_{\lambda\mu} - I \right) \quad , \tag{14}$$

where γ is a small positive constant and the maps $c^i_{\lambda\mu}$ are defined by

$$c^1_{\lambda\mu} : \begin{cases} \omega_{\lambda\mu} & \longmapsto \omega_{\lambda\mu} - \gamma S_{\lambda\mu}, \\ \omega_{\lambda+1,\mu} & \longmapsto \omega_{\lambda+1,\mu} + \gamma S_{\lambda\mu}, \end{cases} \tag{15}$$

$$c^2_{\lambda\mu} : \begin{cases} \omega_{\lambda\mu} & \longmapsto \omega_{\lambda\mu} - \gamma T_{\lambda\mu}, \\ \omega_{\lambda,\mu+1} & \longmapsto \omega_{\lambda,\mu+1} + \gamma T_{\lambda\mu}. \end{cases} \tag{16}$$

Above we have used the definitions

$$S_{\lambda\mu} = N^{\lambda\mu} + N^{\lambda+1,\mu} \ , \quad N^{\lambda\mu} = \omega_{\lambda\mu} d_2 \psi_{\lambda\mu}, \tag{17}$$

$$T_{\lambda\mu} = M^{\lambda\mu} + M^{\lambda,\mu+1} \ , \quad M^{\lambda\mu} = -\omega_{\lambda\mu} d_1 \psi_{\lambda\mu} \quad . \tag{18}$$

Finally, the operator $\mathcal{A}_p$ represents a multivariate Poisson-type process which describes the constraint connecting the stream function and the vorticity. This constraint is implemented by adding the dynamical equation $\dot{\psi} = (\Delta\psi + \omega)/\varepsilon$. On letting $\varepsilon \longrightarrow 0$, the stream function ψ becomes a fast variable (enslaved by the slow variable ω) which is confined to the vicinity of the constraint manifold given by $\Delta\psi + \omega = 0$ (for details see Ref. [5]). $\mathcal{A}_p$ is defined by

$$\mathcal{A}_p = \frac{1}{\beta\varepsilon} \sum_{\lambda\mu} \left(\det[(d_{\lambda\mu})^{-1}] \mathfrak{d}_{\lambda\mu} - I \right) \quad , \tag{19}$$

where β is a small positive constant and

$$d_{\lambda\mu} : \psi_{\lambda\mu} \longmapsto \psi_{\lambda\mu} + \beta W_{\lambda\mu} \ , \quad \beta = \alpha \delta l^2 \quad , \tag{20}$$

and

$$W_{\lambda\mu} \equiv D\psi_{\lambda\mu} + g\omega_{\lambda\mu} \ , \quad g \equiv 1 - \varepsilon\nu \quad . \tag{21}$$

Having defined the stochastic process (ω, ψ) by the above master equation, it is now easy to derive the dynamic equations for arbitrary moments and correlation functions of the stochastic variables. This can be done most easily by deriving the equation of motion of the multivariate characteristic function which is defined by the expectation value

$$\Phi(j, z, t) \equiv \left\langle \exp\left\{ i\delta l^2 \sum_{\lambda\mu} (j_{\lambda\mu}\omega_{\lambda\mu} + z_{\lambda\mu}\psi_{\lambda\mu}) \right\} \right\rangle . \tag{22}$$

Note that the characteristic function $\Phi(j, z, t)$ depends on the two $(n+1)^2$-dimensional arrays $j \equiv \{j_{\lambda\mu}\}$ and $z \equiv \{z_{\lambda\mu}\}$ and that the partial derivatives with respect to $j_{\lambda\mu}$ and $z_{\lambda\mu}$ generate the various correlation functions of the underlying stochastic process. For example, we have

$$\langle \omega_{\lambda\mu}\omega_{\lambda'\mu'} \cdots \rangle = \left(\frac{1}{i\delta l^2} \frac{\partial}{\partial j_{\lambda\mu}} \right) \times \left(\frac{1}{i\delta l^2} \frac{\partial}{\partial j_{\lambda'\mu'}} \right) \cdots \Phi(j, z, t) \Big|_{j=z=0} \quad . \tag{23}$$

The time evolution equation for $\Phi(j, z, t)$ is immediately obtained by using the general definition (5) for the expectation value, differentiating Eq. (22) with respect to time t, and invoking the master equation (7).

The equation of motion for the generating function Φ then allows the comparison with the Hopf functional equation [4] of statistical fluid mechanics mentioned at the beginning. This is done by performing first an asymptotic expansion in $\alpha \longrightarrow 0$ and by taking then the limit of continuous space $\delta l \longrightarrow 0$. In the continuum limit the multivariate characteristic function $\Phi(j, z, t)$ becomes a functional $\Phi[j, z, t]$ in the space of functions $j(\vec{x})$ and $z(\vec{x})$, whereas partial derivatives with respect to $j_{\lambda\mu}$ and $z_{\lambda\mu}$ turn into the corresponding functional derivatives. Following the procedure in Ref. [5], we obtain including terms of order $\mathcal{O}(\alpha\delta l^2)$,

$$\begin{aligned} \frac{\partial \Phi(j, z, t)}{\partial t} = {} & i \int dxdy \left[j(\vec{x}) - \Delta^{-1} z(\vec{x}) \right] \\ & \times \Bigg(\nu\Delta \frac{\delta}{i\delta j} + \left[\frac{\partial}{\partial y} \frac{\delta}{i\delta j} \frac{\partial}{\partial x} \frac{\delta}{i\delta z} - \frac{\partial}{\partial x} \frac{\delta}{i\delta j} \frac{\partial}{\partial y} \frac{\delta}{i\delta z} \right] \Bigg) \Phi(j, z, t) \\ & - \nu\alpha\delta l^2 \int dxdy \left| \vec{\nabla} j \right|^2 \frac{\delta^2}{i^2 \delta j^2} \Phi(j, z, t) \quad . \end{aligned} \tag{24}$$

As one can see from this equation, the dominant contribution which is independent of the parameters α, β, γ, and δl is precisely of the form of the Hopf functional equation in the two-dimensional stream function formulation. Thus, we conclude that within the continuum limit the stochastic process defined by the multivariate master equation (7) yields a complete description of the stochastic properties of the turbulent fields. In particular, the whole hierarchy of dynamical equations for the n-point turbulent correlation functions is correctly described.

The second term on the right-hand side in Eq. (24) is a functional which obviously vanishes in the continuum limit $\alpha \longrightarrow 0$. The second functional derivative of Eq. (24) taken at $j = z = 0$ leads to the dynamic equation for the two-point vorticity correlation function. On Fourier transforming and assuming spatial homogeneity, we obtain

$$\left(\frac{\partial}{\partial t} + 2\nu \vec{k}^2 \right) \langle \omega^*_{\vec{k}} \omega_{\vec{k}} \rangle = W_{\vec{k}} + 2\nu \vec{k}^2 \sigma^2 \quad , \tag{25}$$

where the Fourier transform of a function $f(\vec{x})$ is defined by $f_{\vec{k}} = \int dxdy f(\vec{x}) \exp(-i\vec{k} \cdot \vec{x})$ and $W_{\vec{k}}$ denotes the vorticity transfer function defined by

$$W_{\vec{k}} := \frac{1}{L^2} \sum_{\vec{q}} (k_y q_x - k_x q_y) \langle \omega^*_{\vec{k}} \, \psi_{\vec{q}} \, \omega_{\vec{k}-\vec{q}} \rangle + \text{c.c.} \quad , \tag{26}$$

and $\sigma^2 = \alpha \langle \omega^2 \rangle \delta l^2 L^2$. The second term on the right-hand side in Eq. (25) is the Fourier transform of the correlation function $S(\vec{x}_1, \vec{x}_2)$ of a random vorticity stress which is given by the second functional derivative of the last term in Eq. (24):

$$S(\vec{x}_2, \vec{x}_1) = -2\nu L^{-2}\vec{\nabla}_1\sigma^2\vec{\nabla}_1\delta(\vec{x}_2 - \vec{x}_1) \quad . \tag{27}$$

The stationary solution of Eq. (25) is given by $\langle\omega^*_{\vec{k}}\omega_{\vec{k}}\rangle_{\rm stat} = \sigma^2$. Thus, the spontaneous fluctuations induced by the random vorticity stress are approximately related to the fluctuations occurring in the equilibrium distribution $P_{\rm eq} \sim \exp(-\Omega/k_BT)$ provided we identify

$$\alpha = 2\frac{f k_B T_\Omega}{\langle\omega^2\rangle} \quad . \tag{28}$$

Here Ω denotes the enstrophy (total mean square vorticity) as functional of the vorticity field, T_Ω is the corresponding enstrophy temperature [10,11] and k_B the Boltzmann constant. Furthermore, $f = (L/\delta l)^2$ denotes the number of degrees of freedom. Thus, we conclude that the parameter α introduced into the master equation is equal to two times the ratio of the enstrophy which is contained in the thermal degrees of freedom to the total enstrophy of the field. Once the parameter α is fixed, the parameter γ can be chosen, for example, in such a way that the fluctuations induced by the convection operator $\mathcal{A}_c$ are of the same order of magnitude as the fluctuations given by S.

It is important to note that the form of the random stress is known explicitly from Eq. (24). It is therefore possible to disentangle the thermodynamic from the hydrodynamic degrees of freedom and to obtain the zero-temperature quantities by a kind of renormalization procedure which is explained in detail in Ref. [5]. This renormalization procedure has been used in the analysis of the stochastic simulation data to be discussed in the next section.

III. STOCHASTIC SIMULATIONS OF THE MASTER EQUATION

The stochastic simulation technique which allows us to generate realizations of the stochastic process defined by a master equation of the type (7) is explained in detail in Ref. [5]. In the following we report on three different runs denoted by $R1$, $R2$, and $R3$, respectively. In all cases we choose $L = 1$ and the initial vorticity field is given by a Fourier series of statistically independent modes each of which is drawn from a Gaussian ensemble in such a way that the initial spectrum is given by $\langle\omega^*_{\vec{k}}\omega_{\vec{k}}\rangle_{t=0} = Ak^2\exp(-k/4\pi)$. Here, $\omega_{\vec{k}}$ denotes the Fourier transform of the vorticity field and $A = 1/768\pi^3$. We use the Reynolds numbers [12] ${\rm Re} = \sqrt{2E}L/\nu$, $R_L = E/\nu(2\nu P)^{1/3}$ and $R_\lambda = \Omega^{3/2}/2\nu P$, where $E = \langle\vec{v}^2\rangle/2$ and Ω and P denote the enstrophy, and the palinstrophy (the enstrophy dissipation rate ε_ω divided by 2ν), respectively. The simulation parameters are $\alpha = 0.02$, $\beta = 2.5\times10^{-6}$, $\gamma = 0.02$, $\varepsilon = 10$ for runs $R1$ and $R2$, and $\alpha = 0.07$, $\beta = 6.25\times10^{-7}$, $\gamma = 0.02$, $\varepsilon = 50$ for run $R3$. The physical parameters are summarized in Table I. In all cases the simulation has been performed from $t = 0$ to 25, where one time unit corresponds to approximately one turnover time of the large eddies.

TABLE I. Initial values of the physical parameters of the stochastic simulations. N_r denotes the number of realizations used.

Run	N_r	Resolution	10^3E	Ω	P	Re	R_L	R_λ
$R1$	5	256^2	0.5472	0.5067	1560	29773	3253	104.0
$R2$	5	256^2	0.5077	0.4799	1533	57360	7653	195.2
$R3$	3	512^2	0.6130	0.5395	1667	252097	57045	855.6

First, we show in Fig. 1 the palinstrophy as a function of time for the three runs. After a sharp rise indicating strong nonlinear interaction the palinstrophy reaches a maximum and then decays. As is to be expected the higher the Reynolds number the higher is the time that corresponds to the maximum enstrophy dissipation and the higher is the value of that maximum.

Let us discuss in some detail the behavior of the energy spectrum. To this end, we first illustrate the renormalization procedure mentioned in Sec. II. Recall that the structure of the random vorticity stress is explicitly known from the expansion leading to the Hopf functional equation. This fact allows us to substract the thermal fluctuations from the simulation data in order to obtain the statistical quantities at zero enstrophy temperature $T_\Omega = 0$, that is the "bare" quantities. To this end, we decompose the vorticity field into two parts,

$$\omega_{\vec{k}} = \hat{\omega}_{\vec{k}} + \eta_{\vec{k}} \quad , \tag{29}$$

in such a way that the first part which is denoted by $\hat{\omega}$ obeys the equation

$$\left(\frac{\partial}{\partial t} + 2\nu\vec{k}^2\right)\langle\hat{\omega}^*_{\vec{k}}\hat{\omega}_{\vec{k}}\rangle_t = \hat{W}_{\vec{k}} \quad , \tag{30}$$

where $\hat{W}_{\vec{k}}$ is defined as in Eq. (26) with $\omega_{\vec{k}}$ replaced by $\hat{\omega}_{\vec{k}}$. In accordance with our above discussion $\hat{\omega}$ represents the vorticity field at zero enstrophy temperature. The second part $\eta_{\vec{k}}$ in Eq. (29) denotes a random field which is statistically independent from the vorticity and which obeys

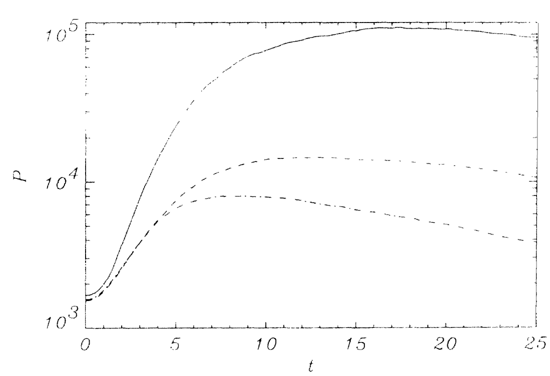

FIG. 1. The palinstrophy P as a function of time for the three stochastic simulations $R1$ (dashed-dotted line), $R2$ (dashed line), and $R3$ (continuous line).

$$\langle \eta_{\vec{k}} \rangle_t = 0, \quad \langle \eta^*_{\vec{k}} \eta_{\vec{k}} \rangle_t = g_{\vec{k}}(t) \quad . \tag{31}$$

Consequently, we have

$$\langle \omega^*_{\vec{k}} \omega_{\vec{k}} \rangle_t = \langle \hat{\omega}^*_{\vec{k}} \hat{\omega}_{\vec{k}} \rangle_t + g_{\vec{k}}(t) \quad . \tag{32}$$

Inserting Eq. (32) into Eq. (25) and defining the correlation function of the random field $\eta_{\vec{k}}$ by

$$g_{\vec{k}}(t) = 2\nu \vec{k}^2 \int_0^t d\tau \, \sigma^2(\tau) \exp\left\{-2\nu \vec{k}^2 (t-\tau)\right\}, \tag{33}$$

we obtain Eq. (30). Thus we conclude that the noise part in Eq. (25) induced by the random vorticity stress is removed by the simple transformation (32) which may therefore be used in order to separate uniquely the zero-temperature field from the random vorticity governed by our master equation. It should be clear that the above transformation removes the thermal noise only on the level of the two-point correlation functions and that the influence of the random stresses upon the probabilistic dynamics is left unchanged.

We shall illustrate this renormalization procedure by means of our simulation data from run $R3$. To this end, we first depict in Fig. 2 the energy spectrum:

$$E_k = \frac{1}{2L^2 k} \langle \omega^*_{\vec{k}} \omega_{\vec{k}} \rangle_t \tag{34}$$

as it is obtained directly from our stochastic simulation by averaging over the three realizations and over 1 turnover time around $t = 21.75$ (see the continuous line in Fig. 2). The dashed-dotted line in Fig. 2 represents the function $g_{\vec{k}}(t)/2L^2k$ (at the same time) estimated again from our simulation data (the expression for the random vorticity stress has been determined to one higher order in δl than was done above). Figure 3 shows the difference of both curves, that is the renormalized zero-temperature energy spectrum

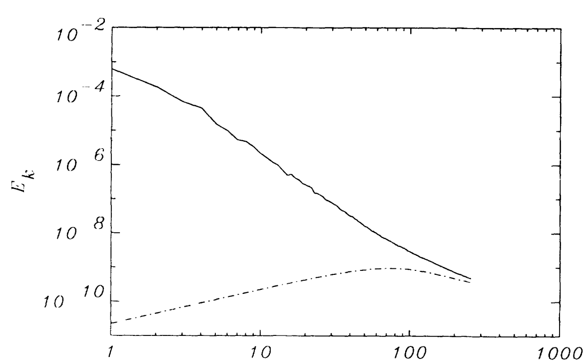

FIG. 2. Continuous line: Log-log plot of the energy spectrum E_k [see Eq. (34)] as it is obtained directly from the simulation data of run $R3$ by averaging over three realizations and over one turnover time around the time $t = 21.75$. Dashed-dotted line: The spectrum $g_{\vec{k}}$ of the correlation function of the random vorticity stress divided by $2L^2k$ for the same run at the same time.

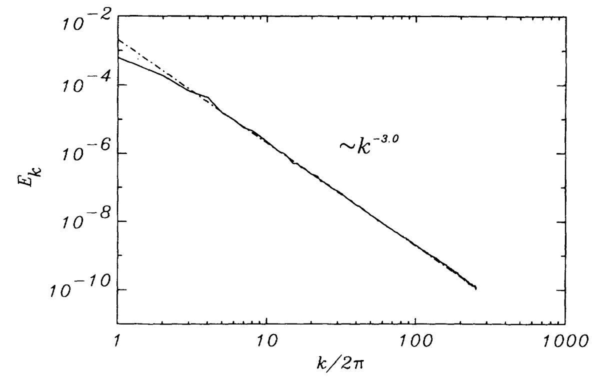

FIG. 3. Continuous line: Log-log plot of the renormalized energy spectrum $\hat{E}_k$ [see Eq. (35)] obtained as the difference of the curves of Fig. 2. Dashed-dotted line: The least square fit of the data to a straight line giving the spectral exponent -3.0 ± 0.1.

$$\hat{E}_k = \frac{1}{2L^2 k} \langle \hat{\omega}^*_{\vec{k}} \hat{\omega}_{\vec{k}} \rangle_t = \frac{1}{2L^2 k} \left(\langle \omega^*_{\vec{k}} \omega_{\vec{k}} \rangle_t - g_{\vec{k}} \right) \tag{35}$$

together with a least square fit within the range $5 \leq k/2\pi \leq 80$. Figure 3 clearly demonstrates that the energy spectrum in the inertial range nicely fits to a power law behavior of the form

$$\hat{E}_k = C k^{-m} \, , \quad C = \text{const.} \tag{36}$$

The least square fit gives for the spectral exponent $m = -3.0 \pm 0.1$ which is in perfect agreement with the prediction of the Kraichnan-Batchelor theory [6,7] of the vorticity cascade in 2D turbulence and is also consistent with experimental measurements [13] and other numerical methods [14,15]. The error indicated above has been estimated by choosing different intervals within which the fit is perfomed.

In order to give an impression of how this asymptotic value of the spectral exponent is reached, we plot in Fig. 4 the spectral exponent m for the three different Reynolds numbers of our three runs as a function of time. The

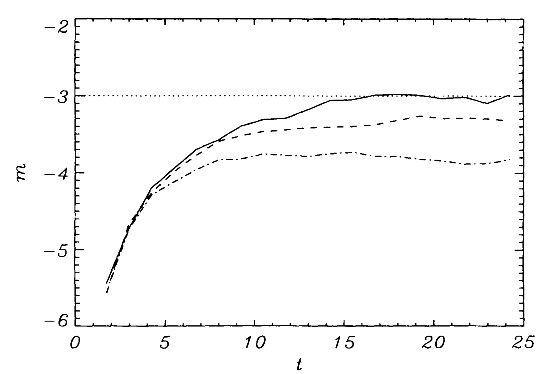

FIG. 4. The spectral exponent m [see Eq. (36)] obtained by least square fits to the renormalized spectra as a function of time for the three stochastic simulations $R1$ (dashed-dotted line), $R2$ (dashed line), and $R3$ (continuous line).

exponent m has been estimated from the renormalized energy spectra by least square fits within the range $10 \leq k/2\pi \leq 60$ for $R1$ and $R2$ and within the range $5 \leq k/2\pi \leq 80$ for run $R3$. The Reynolds number R_L for run $R2$ is 2.3 times higher than for run $R1$ and it is 7.4 times higher for run $R3$ than for run $R2$. Thus, Fig. 4 clearly shows that the asymptotic value $m = -3$ of the Kraichnan-Batchelor theory is indeed reached for very high Reynolds numbers after a certain transient time. Note that we do not observe any hints of a transition of the spectral exponent from -4 to -3 [14] nor do we see anything special about the exponent $25/7 \approx 3.57$ as has been speculated by Polyakov [16] on the basis of a theory employing methods from conformal field theory.

According to the dimensional analysis of the Kraichnan-Batchelor theory the k-independent constant C in Eq. (36) is given by

$$C = C_{\rm KB}/\varepsilon_\omega^{2/3} \quad , \tag{37}$$

where ε_ω is the enstrophy dissipation rate. We estimate from the data of run $R3$ that the dimensionless constant $C_{\rm KB}$ is about 0.5.

IV. CONCLUSIONS

An appropriate description of the problem of turbulence often requires in addition to the Navier-Stokes equation a statistical approach in terms of ensembles. For example, the Kraichnan-Batchelor theory of 2D turbulence is an application of the Kolmogorov theory of 3D turbulence to the idea of a vorticity cascade. Thus, as in the Kolmogorov theory the predictions of the Kraichnan-Batchelor theory are based upon statistical arguments in an essential way. In particular, the notion of universality is a consequence of the assumption of statistical independence of the small scale motion from the large eddies and is thus intimately connected to a statistical description of the flow field.

As has been mentioned in the Introduction, our master equation approach introduces one further stochastic element since the time evolution itself is also made intrinsically probabilistic. Therefore, the realizations of the Markov process defined by our master equation are not smooth and do not obey the Navier-Stokes equation. However, as has been demonstrated in Sec. II, the equation of motion of the generating functional of the stochastic process (ω, ψ) defined by the master equation leads in the continuum limit to the Hopf functional equation of the two-dimensional turbulence. This means that within the continuum limit the total hierarchy of moment equations is correctly described by the stochastic process defined by our master equation.

Furthermore, the probabilistic character of the time-evolution gives rise to an additional term in the equation of motion for the generating functional which reflects the presence of random stresses in the equation for the two-point vorticity correlation function. The explicit form of the spectrum of these random stresses is known from the asymptotic expansion leading to the Hopf functional equation. It is thus possible to subtract the energy content of the thermal degrees of freedom from the simulation data in order to obtain the energy spectrum at zero temperature. This renormalization procedure has been explained in detail in Sec. III and has been illustrated by means of our simulation data.

It is important to emphasize the differences of our approach to other methods of simulating two-dimensional turbulence in which random noise is present. For example, in contrast to lattice-gas simulations [17], our master equation formulation is a true mesoscopic one since it involves the same set of dynamic variables as the hydrodynamic macroscopic description. Within direct numerical simulations with spectral codes, one often includes the effect of random stirring forces (see, e.g., Ref. [18]). These random forces represent an *external* source of noise the statistical properties of which (e.g., the correlation spectrum) are assumed to be given. The random noise contained in our master equation description, however, is to be referred to as *internal* noise which results from the thermal fluctuations of the hydrodynamic variables.

The stochastic simulations presented in Sec. III exhibit basic properties which are known from numerical and theoretical investigations of two-dimensional turbulence. In particular, the simulation results clearly demonstrate the existence of a k^{-3} power law in the inertial range of the energy spectrum for the run with Reynolds number $\mathrm{Re} = 2.5 \times 10^5$ ($R_L = 5.7 \times 10^4$). We conclude that the stochastic simulation technique based on our master equation is a suitable and powerful method for the estimation of statistical quantities of the turbulent flow field. In particular, it might be important to note that the stochastic simulation method suggested in this paper leads to numerical algorithms which can easily be implemented on a parallel processor [19].

[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence* (MIT Press, Cambridge, MA, 1981), Vol. 2.

[2] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[3] H. P. Breuer and F. Petruccione, Phys. Rev. E **47**, 1803 (1993).

[4] E. Hopf, J. Ration. Mech. Anal. **1**, 87 (1952).

[5] H. P. Breuer and F. Petruccione, J. Phys. A **26**, 7563 (1993).

[6] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).

[7] G. K. Batchelor, Phys. Fluids Suppl. **II**, 233 (1969).

[8] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, London, 1959).

[9] H. P. Breuer and F. Petruccione, Phys. Lett. A **185**, 385 (1994).

[10] R. H. Kraichnan and D. Montgomery, Rep. Prog. Phys. **43**, 547 (1980).

[11] H. A. Rose and P. L. Sulem, J. Phys. (Paris) **39**, 441 (1978).
[12] J. R. Herring, S. A. Orszag, R. H. Kraichnan, and D. G. Fox, J. Fluid Mech. **66**, 417 (1974).
[13] M. Gharib and P. Derango, Physica D **37**, 406 (1989).
[14] M. E. Brachet, M. Meneguzzi, H. Politano, and P. L. Sulem, J. Fluid Mech. **194**, 333 (1988).
[15] P. Santangelo, R. Benzi, and B. Legras, Phys. Fluids A **1**, 1027 (1989).
[16] A. M. Polyakov, Nucl. Phys. B **396**, 367 (1993).
[17] S. Succi, P. Santangelo, and R. Benzi, Phys. Rev. Lett. **60**, 2738 (1988).
[18] V. Borue, Phys. Rev. Lett. **72**, 1475 (1994).
[19] H. P. Breuer and F. Petruccione, Freiburg Report No. THEP 93/29, 1993 (unpublished).